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Inequality and Comparison Theory for Non-linear Differential and Integral Equations

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Abstract : The present work deals with some basic outcomes related to strict and non-strict non-linear differential and integral inequalities. Further the comparison result is proved and used to discuss the uniqueness theorem.

Index Terms: Non-linear differential equations, Strict and Non-Strict Inequalities, Comparison principle, Existence theorem.

1. INTRODUCTION .

Let R be real line which is connected set, i.e., which cannot be written in the form of union of two separated sets. Given a bounded interval $J=[t_0,t_0+a]$ in R for some fixed $t_0,a\in R$ *with* $a>0$.

Consider the initial value problems for non-linear differential equations

$$\frac{d}{dt}[x(t)-f(t,x(\alpha(t)))]=g(t,x(\alpha(t))),\ t\in J$$

$$x(t_0)=x_0\in R \quad and \quad \alpha(t)\in R \qquad (1.1)$$

Where $f,g:J\times R\to R$ are continuous real valued functions defined on J.

By a solution of the (1.1), we mean a function $x\in C(J,R)$ such that

i) the function $t\to x-f(t,x(\alpha(t)))$ is continuous for each $x\in R$ and $\alpha(t)$ be any scalar valued function.

ii) x satisfies the equation (1.1).

The importance of the investigations of the work of non-linear differential equations lies in the fact that they include various dynamic systems as special cases.[6,15,16] The consideration of non-linear differential equations is implicit in the work of Krasnoselskii [13] and extensively treated in the various research papers on non-linear differential equations with different perturbations. see Krasnoselskii[13] and references therein. This class of differential and integral equations includes the perturbation of original differential and integral equations in different ways [7,8,9,10,11,12].

In this paper, we initiate the basic theory of non-linear differential equation's of mixed inequalities and existence theorem. Our claim is that, the outcomes of this paper are of basic level and significant contribution to the theory of non-linear ordinary differential equations.

2. STRICT AND NON-STRICT INEQUALITIES

We need frequently the following hypothesis in what follows:

(A_0) The function $x\to x-f(t,x(\alpha(t)))$ is increasing in R for all $t\in J$.

We begin by proving the basic results dealing with non-linear differential inequalities.

Theorem 2.1: Assume that the hypothesis (A_0) holds. Suppose that there exist $y,z\in C(J,R)$

such that $$\frac{d}{dt}[y(t)-f(t,y(\alpha(t)))]\le g[t,y(\alpha(t))],\ t\in J \qquad (2.1)$$

and $$\frac{d}{dt}[z(t)-f(t,z(\alpha(t)))]\ge g[t,z(\alpha(t))],\ t\in J \qquad (2.2)$$

If one of the inequalities (2.1) and (2.2) is strict and

$$y(t_0)<z(t_0) \qquad (2.3)$$

then $$y(t)<z(t) \qquad (2.4)$$

for all $t\in J$.

Proof :-Suppose that the inequality (2.4) is false, then the set Z defined by

$$Z=\{t\in J: y(t)\ge z(t)\} \qquad (2.5)$$

Is non empty.

Denote $t_1=\inf Z$ without loss of generality, we may assume that

$$y(t_1)=z(t_1) \quad and \quad y(t)<z(t) \quad for\ all\ t<t_1.$$

Assume that $\frac{d}{dt}\left[z(t)-f(t,z(\alpha(t)))\right] > g(t,z(\alpha(t)))$ *for all* $t \in J$.

Denote

$$Y(t)=\left[y(t)-f(t,y(\alpha(t)))\right] \text{ and}$$

$$Z(t)=\left[z(t)-f(t,z(\alpha(t)))\right] \text{ for all } t \in J.$$

Now continuity of y and z together with (2.3) implies that there exists a $t_1 > t_0$ *such that*

$$y(t_1)=z(t_1) \text{ and } y(t)<z(t) \qquad (2.6)$$

For all $t_0 \le t \le t_1$.

As (A_0) holds, it follows from (2.5) that

$$Y(t_1)=y(t_1)-f(t_1,y(\alpha(t_1)))$$
$$=z(t_1)-f(t_1,z(\alpha(t_1)))=Z(t_1).$$

and

$$Y(t)=y(t)-f(t,y(\alpha(t)))$$
$$<z(t)-f(t,z(\alpha(t)))$$
$$\Rightarrow Y(t)<Z(t) \qquad (2.7) \text{ for all } t_0 \le t < t_1.$$

From the above relation (2.7) , we obtain

$$Y(t_1+h)=Z(t_1+h) \text{ and } Y(t_1)<Z(t_1)$$
$$\Rightarrow -Y(t_1)>-Z(t_1)$$
$$\therefore Y(t_1+h)-Y(t_1)>Z(t_1+h)-Z(t_1)$$

Dividing both sides by $h \neq 0$

$$\frac{Y(t_1+h)-Y(t_1)}{h} > \frac{Z(t_1+h)-Z(t_1)}{h}$$

For small h < 0.

Taking the limit as $h \to 0$, we obtain

$$Y'(t_1) \ge Z'(t_1) \qquad (2.8)$$

Hence from inequality (2.7) and (2.8) , we get

$$g(t_1,y(\alpha(t_1))) \ge Y'(t_1) \ge Z'(t_1) > g(t_1,z(\alpha(t_1)))$$

Which is a contradiction o our assumption that $y(t) \ge z(t)$.

Hence $y(t)<z(t)$ *for all* $t \in J$.

In the next theorem we discuss the non-strict inequality for the non-linear differential equation (1.1) on J in which one sided Lipschitz condition used.

Theorem 2.2 :- Assume that the hypothesis of theorem 2.1 hold. Suppose also that there exists a real number L>0 such that

$$g(t,y(\alpha(t)))-g(t,z(\alpha(t))) \le L \underset{t_0<r<t}{Sup}\left[y(r)-f(r,y(\alpha(r)))-(z(r)-f(r,z(\alpha(r))))\right] \qquad (2.9)$$

Whenever $y(r) \ge z(r)$, $t_0 \le r < t$. then

$$y(t_0) \le z(t_0) \qquad (2.10)$$

$$\Rightarrow y(t) \le z(t) \qquad (2.11)$$

for all $t \in J$.

Proof:- Let $\in > 0$ be given and let L>0 be any given real number. Set define

$$z_\in(t)-f(t,z_\in(\alpha(t)))=z(t)-f(t,x(\alpha(t)))+\in e^{2P(t-t_0)} \qquad (2.12)$$

so that

$$z_\in(t)-f(t,z_\in(\alpha(t))) > z(t)-f(t,x(\alpha(t)))$$

We define

$$Z_\in(t)=z_\in(t)-f(t,z_\in(\alpha(t))) \text{ and } Z(t)=z(t)-f(t,z(\alpha(t))) \text{ for all } t \in J.$$

Now using inequality (2.9), we have

$$g(t,z_\in(\alpha(t)))-g(t,z(\alpha(t))) \le L \underset{t_0 \le r<t}{Sup}\left[Z_\in(r)-Z(r)\right]$$
$$=L \in e^{2L(t-t_0)}$$

Now ,

$$Z'_{\epsilon}(t) = Z(t) + 2L \in e^{2L(t-t_0)}$$
$$\geq g(t, z(t)) + 2L \in e^{2L(t-t_0)}$$
$$\geq g(t, z_{\epsilon}(t)) + 2L \in e^{2L(t-t_0)} - L \in e^{2L(t-t_0)}$$
$$\Rightarrow Z'_{\epsilon}(t) \geq g(t, z_{\epsilon}(t)) \quad for\ all\ t \in J.$$

Also we have $Z_{\epsilon}(t_0) > Z(t_0) \geq Y(t_0).$ *for all* $t \in J.$

Now using theorem 2.1 with $z = z_{\epsilon}$, to give

$$Y(t) < Z_{\epsilon}(t) \quad for\ all\ t \in J.$$

On taking $\epsilon \to 0$ *in the above inequalty,we get* $Y(t) \leq Z(t)$

Which is further in view of hypothesis (A_0) implies that (2.11) holds on J. Hence the proof.

3. EXISTENCE RESULT

In this article we prove an existence result for the non-linear differential equation (1.1) On a closed and bounded interval $J = [t_0, t_0 + a]$ under the mixed Lipschitz and Compactness Conditions on the non-linearity involved in it.

We use the non-linear differential equation (1.1) in the space C(J,R) of continuous real valued Functions defined on $[t_0, t_0 + a]$

In C(J,R) we defIn C(J,R) we define a $\|\square\|$ as $\|x\| = \sup_{t \in J} |x(t)|$. Clearly C(J,R) is a separable Banach space with respect to the above supremum norm. We prove The existence of solutions for the non-linear differential equation (1.1) via the following fixed point theorem in the Banach spaces.[4]

Theorem 3.1 Suppose that S is closed, convex and bounded subset of the separable Banach space E and let $A : E \to E$ and $B : S \to E$ be two operators such that

a) A is non-linear contraction

b) B is compact and continuous, and

c) $x = Ax + By$ *for all* $y \in S \Rightarrow x \in S$.

Then the operator equation $Ax + By = x$ has a solution in S.

We consider the following hypothesis in what follows.

(A_1) There exists a constant L>0 such that $|f(t,x) - f(t,y)| \leq \dfrac{L|x-y|}{M + |x-y|}$,

for all $t \in J$ *and* $x, y \in R.$ *moreover* $L \leq M.$

(A_2) There exists a continuous function $h : J \to R$ such that $|g(t,x)| \leq h(t)$ $t \in J$, *for all* $x \in R.$

To prove the theorem the following lemma is useful which is discussed in sequel.

Lemma 3.1 Assume that hypothesis (A0) holds. Then for any continuous function $h : J \to R$, the function $x \in C(J,R)$ is a solution of non-linear differential equation

$$\frac{d}{dt}[x(t) - f(t, x(\alpha(t)))] = h(t) \quad for\ all\ t \in J, \tag{3.1}$$
$$x(0) = x_0 \in R$$

If and only if x satisfies the non-linear differential equation

$$x(t) = x_0 - f(t_0, x(\alpha(t_0))) + f(t, x(\alpha(t))) + \int_{t_0}^{t} h(s)ds \tag{3.2}$$

Proof :- Let $h \in C(J,R)$

We first assume that x satisfy the (3.1) then by definition $x(t) - f(t, x(\alpha(t)))$ is continuous on the interval $J = [t_0, t_0 + a)$ and so it is differentiable there, as a result $\dfrac{d}{dt}[x(t) - f(t, x(\alpha(t)))]$ is integrable on J.

Integrating (3.1) from t_0 to t, we have

$$\int_{t_0}^{t} \frac{d}{dt}[x(t) - f(t, x(\alpha(t)))]dt = \int_{t_0}^{t} h(t)dt$$

$$[x(t) - f(t, x(\alpha(t)))]_{t_0}^{t} = \int_{t_0}^{t} h(s)ds$$

$$x(t_0) = x_0$$

$$i.e.,\left[x(t)-f(t,x(\alpha(t)))\right]=\left[x(t_0)-f(t_0,x(\alpha(t_0)))\right]+\int_{t_0}^{t}h(s)ds\,,\ t\in J$$

$$\therefore x(t)=x(t_0)-f(t_0,x(\alpha(t_0)))+f(t,x(\alpha(t)))+\int_{t_0}^{t}h(s)ds\ \ t\in J.$$

Conversely suppose that x satisfies

$$x(t)=x(t_0)-f(t_0,x(\alpha(t_0)))+f(t,x(\alpha(t)))+\int_{t_0}^{t}h(s)ds\ \ t\in J.$$

Differentiating above equation we get $\frac{d}{dt}\left[x(t)-f(t,x(\alpha(t)))\right]=h(t),\ \ t\in J$

Now substituting $t=t_0$ in (3.2) ,we get

$$x(t_0)=x_0-f(t_0,x(\alpha(t_0)))+f(t_0,x(\alpha(t_0)))$$

$$\therefore x(t_0)-f(t_0,x(\alpha(t_0)))=x_0-f(t_0,x(\alpha(t_0)))$$

Since the mappiSi $x\mapsto x-f(t($ is an increasing in R for all $J\in R$. Also the mapping $x\mapsto x-f(t_0$ is one one in R. This proves $x(t_0)=x_0$ .This completes the lemma.

Now we are going to discuss the following existence theorem for the Non-linear differential equation (1.1) on the interval J.

Theorem 3.2 : Assume that the hypothesis $(A_0)-(A_2)$ hold. Then the non-linear differential equation (1.1) has a solution defined on J.

Proof : Let set E=C(J,R) and define a subset S of E defined by

$$S=\{x\in E\,|\,\|x\|\le N\} \qquad (3.3)$$

Where $N=|x_0-f(t_0,x(\alpha(t_0)))|+L+F_0+\|h\|$

$F_0>0\ \ such\ that\ F_0=\sup_{t\in J}|f(t,x(\alpha(t)))|$

Clearly S is a closed , convex and bounded subset of the Banach space E.

Now using the hypothesis (A_0) and (A_2) and application of lemma 3.1, we can easily show that the non-linear differential equation (1.1) is equivalent to the non-linear integral equation

$$x(t)=x_0-f(t_0,x(\alpha(t_0)))+f(t,x(\alpha(t)))+\int_{t_0}^{t}g(s,x(\alpha(s)))ds \qquad (3.4)$$

for $t\in J$.

We define two operators $A:E\to E\ \ and\ \ B:S\to E$ by

$$Ax(t)=f(t,x(\alpha(t)))\ ,t\in J \qquad (3.5)$$

And $$Bx(t)=x_0-f(t_0,x(\alpha(t_0)))+\int_{t_0}^{t}g(s,x(\alpha(s)))ds\ \ ,t\in J \qquad (3.6)$$

Then the integral equation (3.5) is transformed into an operator equation as

$$Ax(t)+Bx(t)=x(t)\ \ ,t\in J \qquad (3.7)$$

Our aim is to show that the operators A and B satisfy all the conditions of theorem (3.1).

i.e., we first show that A is a Lipschitz operator on E with the Lipschitz constant L.

Let x, y be any two members in E, then by hypothesis (A_1)

$$|A(x(t))-A(y(t))|=|f(t,x(\alpha(t)))-f(t,y(\alpha(t)))|$$

$$\le\frac{L|x(\alpha(t))-y(\alpha(t))|}{M+|x(\alpha(t))-y(\alpha(t))|}$$

$$\le\frac{L\|x-y\|}{M+\|x-y\|}\qquad for\ \ all\ \ \alpha(t)\in R,\ where\ t\in J.$$

This shows that A is a non-linear contraction E with D-function ψ defined by $\psi(r)=\frac{Lr}{M+r}$.

Now we have to show the second condition of theorem (3.1)

i.e., B is compact and continuous operator on S into E.

First we prove, B is continuous on S.

Let $\{x_n\}$ be a sequence in S converging to a point $x\in S$. Then by dominated convergence theorem for integration, we obtain

$$\lim_{n\to\infty} Bx_n(t) = \lim_{n\to\infty}\left[x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^{t} g(s, x_n(\alpha(s)))ds \right]$$

$$= \lim_{n\to\infty} x_0 - \lim_{n\to\infty} f(t_0, x(\alpha(t_0))) + \lim_{n\to\infty} \int_{t_0}^{t} g(s, x_n(\alpha(s)))ds$$

$$= x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^{t} \left[\lim_{n\to\infty} g(s, x_n(\alpha(s))) \right] ds$$

$$= x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^{t} g(s, x(\alpha(s)))ds$$

$$= Bx(t) \ , \quad for \ t \in J.$$

Moreover, it can be shown as below that $\{Bx_n\}$ is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granas et.al[16] , it is proved that B is a continuous operator on S.

Now we have to show B is compact operator on S.

To prove this it is sufficient to show that B(S) is a uniformly bounded and equicontinuous set in E.

Let $x \in S$ be arbitrary. Then by hypothesis (A_2).

$$|Bx(t)| \le |x_0 - f(t_0, x(\alpha(t_0)))| + \int_{t_0}^{t} |g(s, x(\alpha(s)))|ds$$

$$\le |x_0 - f(t_0, x(\alpha(t_0)))| + \int_{t_0}^{t} h(s)ds$$

$$\le |x_0 - f(t_0, x(\alpha(t_0)))| + \|h\| a \quad for \ all \ t \in J.$$

Taking supremum over t,

$$Sup_t |Bx(t)| \le Sup_t \{ |x_0 - f(t_0, x(\alpha(t_0)))| + \|h\| a \}$$

$$\|Bx\| \le |x_0 - f(t_0, x(\alpha(t_0)))| + \|h\| a \ , \ for \ all \ x \in S.$$

This shows that B is uniformly bounded on S. Again let $t_1, t_2 \in J$ .then for any $x \in S$, we have

$$|Bx(t_1) - Bx(t_2)| = \left| \int_{t_0}^{t_1} g(s, x(\alpha(s)))ds - \int_{t_0}^{t_2} g(s, x(\alpha(s)))ds \right|$$

$$\le \left| \int_{t_2}^{t_1} g(s, x(\alpha(s)))ds \right|$$

$$\le |p(t_1) - p(t_2)|$$

Where $p(t) = \int_{t_0}^{t} h(s)ds.$

Since the function p is continuous on compact J, it is uniformly continuous.

Hence, for $\in > 0$, there exists a $\delta > 0$ such that $|t_1 - t_2| < \delta \Rightarrow |Bx(t_1) - Bx(t_2)| < \in$

For all $t_1, t_2 \in J$ and for all $x \in S$.

This shows that B(S) is an equicontinuous set in E.

Now being uniformly bounded and equicontinuous set in E, so it is compact by Arzela-Ascoli theorem. This proves, B is a continuous and compact operator on S.

Now we have to show that $x = Ax + By \ \ for \ all \ y \in S \Rightarrow x \in S \ is \ satisfied.$

Let $x \in E$ and $y \in S$ be arbitrary such that $x = Ax + By$.

Then, by assumption (A_1) , we have

$$|x(t)| = |Ax(t) + By(t)|$$
$$\leq |Ax(t)| + |By(t)|$$
$$\leq |x_0 - f(t_0, x(\alpha(t_0)))| + |f(t, x(\alpha(t)))| + \int_{t_0}^{t} |g(s, y(\alpha(s)))| ds$$
$$\leq |x_0 - f(t_0, x(\alpha(t_0)))| + [|f(t, x(\alpha(t))) - f(t,0)| + |f(t,0)|] + \int_{t_0}^{t} |g(s, y(\alpha(s)))| ds$$
$$\leq |x_0 - f(t_0, x(\alpha(t_0)))| + L + F_0 + \int_{t_0}^{t} h(s) ds$$

Taking supremum over t,

$$\|x\| \leq |x_0 - f(t_0, x(\alpha(t_0)))| + L + F_0 + \|h\| a$$

Thus all the conditions of theorem (3.1) are satisfied and hence the operator equation $Ax + By = x$ has a solution in S.As a result , the non-linear differential equation (1.1) has a solution defined on J .This completes the proof.

4. MAXIMAL AND MINIMAL SOLUTIONS

Under this section we shall discuss the existence of maximal and minimal solutions for the Non-linear differential equation (1.1) on $J = [t_0, t_0 + a]$.

Definition : - A solution of the Non-linear differential equation (1.1) is said to be Maximal if for any other solution x to the Non-linear differential equation (1.1) one has $x(t) \leq r(t)$, *for all* $t \in J$ .Again , a solution ρ of the Non-linear differential equation (1.1) is said to be minimal if $\rho(t) \leq x(t)$, *for all* $t \in J$, where x is any solution of the Non-linear differential equation (1.1) existing on J.

We study the case of Maximal solutions only , as the case of minimal solution is similar and can be proved with the suitable and appropriate modifications.

Given a arbitrary small real number $\in > 0$, consider the following IVP of Non-linear differential equation (1.1)

$$\frac{d}{dt}[x(t) - f(t, x(\alpha(t)))] = g(t, x(\alpha(t)) + \in, t \in J \qquad (4.1)$$
$$x(t_0) = x_0 + \in$$

Where $f, g \in C(J \times R, R)$.

An existence theorem for the Non-linear differential equation (1.1) can be stated as follows:

Theorem 4.1 Assume that the hypothesis (A_0)-(A_2) hold. Then for every small number $\in > 0$, the Non-linear differential equation (1.1) has a solution defined on J.

Our main existence theorem for maximal solution can be stated as

Theorem 4.2 Assume that the hypotheses (A_0)-(A_2) hold. Further $L \leq M$, then the Non-linear differential equation (1.1) has a maximal solution defined on J.

5. COMPARISON THEOREMS

The main use of differential inequalities is to find the bound for the solution set for the differential inequality related to Non-linear differential equation (1.1).In this section we prove that the maximal and minimal solutions serve the bounds for the solutions of the related differential inequality to Non-linear differential equation (1.1) on $J = [t_0, t_0 + a]$.

Theorem 5.1 Assume that the hypotheses (A_0)-(A_2) hold. Further, if there exists a function $u \in C(J, R)$ such that

$$\frac{d}{dt}[u(t) - f(t(u(\alpha(t)))] \leq g(t, u(\alpha(t))), \quad t \in J \qquad (5.1)$$
$$u(t_0) \leq x_0.$$

Then , $u(t) \leq r(t)$ (5.2)

for all $t \in J$,where r is a maximal solution of the Non-linear differential equation (1.1) on J.

Proof:- Let $\in > 0$ be arbitrary small real number. Then by theorem (4.2), $r(t, \in)$ is a Maximal solution of the non-linear differential equation (4.1) and that the limit

$$r(t) = \lim_{\in \to 0} r(t, \in) \qquad (5.3)$$

is uniform on J and the function r is a maximal solution of the non-linear differential equation (1.1) on J. Hence we obtain

$$\frac{d}{dt}[r(t, \in) - f(t, r(\alpha(t), \in))] = g(t, r(\alpha(t), \in)) + \in, t \in J \qquad (5.4)$$
$$r(t_0, \in) = x_0 + \in$$

From above inequality it follows that

$$\frac{d}{dt}\left[r(t,\in)-f(t,r(\alpha(t),\in))\right]>g(t,r(\alpha(t),\in))+\in, t\in J \qquad (5.5)$$

$$r(t_0,\in)>x_0$$

Now we apply theorem (2.1) to the inequalities (5.1) and (5.5) and conclude that

$$u(t)<r(t,\in) \qquad (5.6)$$

For all $t\in J$. This further in view of limit (5.3) implies that inequality (5.2) holds on J. This completes the proof.

Theorem 5.2 Assume that the hypotheses (A_0)-(A_2) hold. Further, if there exists a function $v\in C(J,R)$ such that

$$\frac{d}{dt}\left[v(t)-f(t,v(t)\right]\geq g(t,v(t), t\in J \qquad (5.7)$$

$$v(t_0)\geq x_0.$$

Then, $$\rho(t)\leq v(t) \qquad (5.8)$$

for all $t\in J$, where ρ is a minimal solution of the Non-linear differential equation(1.1)on J.

Note that, Theorem 5.1 is useful to prove the boundedness and uniqueness of the solutions for the Non-linear differential equation (1.1) on J.A result in this direction is

Theorem 5.3 : Assume that the hypotheses (A_0)-(A_2) hold. Suppose that there exists a Function $G: J\times R_+\to R_+$ such that

$$\left|g(t,x_1(\alpha(t))-g(t,x_2(\alpha(t))\right|\leq G\left(t,\max_{s\in[t_0,t]}\left|\left(x_1(s)-f(s,x_1(\alpha(s)))\right)-\left(x_2(s)-f(s,x_2(\alpha(s)))\right)\right|\right) \quad (5.9)$$ for all $x_1,x_2\in E$ and $x_1,x_2\in E$. If identically zero function is the only solution of the differential equation

$$m'(t)=G(t,m(t)),\quad t\in J,\quad m(t_0)=0, \qquad (5.10)$$

then the non-linear differential equation (1.1) has a unique solution defined on J.

Proof:- By theorem 3.2, the non-linear differential equation (1.1) has a solution on J.

Suppose that there are two solutions u_1 *and* u_2 of the non-linear differential equation (1.1) Existing on J. Define a function $m: J\to R_+$ by

$$m(t)=\left|\left(u_1(t)-f(t,u_1(\alpha(t))\right)-\left(u_2(t)-f(t,u_2(\alpha(t))\right)\right|. \qquad (5.11)$$

As $\left(|x(t)|\right)'\leq|x'(t)|$ *for* $t\in J$, we have that

$$\begin{aligned} m'(t) &\leq \left|\frac{d}{dt}\left[u_1(t)-f(t,u_1(\alpha(t)))\right]-\frac{d}{dt}\left[u_2(t)-f(t,u_2(\alpha(t)))\right]\right| \\ &\leq \left|g(t,u_1(t))-g(t,u_2(t))\right| \\ &\leq G\left(t,\left|\left(u_1(t)-f(t,u_1(\alpha(t)))\right)-\left(|u_2(t)-f(t,u_2(\alpha(t)))|\right)\right|\right) \\ &= G(t,m(t)) \end{aligned}$$

for all $t\in J$; *and that* $m(t_0)=0$

Now, we apply Theorem 5.1 with $f\equiv 0$ to get that $m(t)=0$ *for all* $t\in J$.

This gives

$$u_1(t)-f(t,u_1(\alpha(t)))=u_2(t)-f(t,u_2(\alpha(t))) \text{ for all } t\in J.$$

Finally, in view of hypothesis (A0) we conclude that $u_1(t)=u_2(t)$ on J. This completes the proof.

REFERENCES

[1] A. Granas, R.B. Guenther and J.W.Lee, *Some general existence principles for Caratheodory theory of nonlinear differential equation,* J.Math. Pures et Appl.70(1991),153-196.

[2] M.A. Krasnoselskii, Topology Methods in the *Theory of Nonlinear Integral Equations,* Pergamon Press 1964.

[3] T.A. Burton, *A fixed point theorem of Krasnoselskii,* Appl. Math. Lett.11(1998),83-88.

[4] S.Heikkila and V.Lakshmikantham, *Monotone Iterative Technique for Nonlinear Discontinues Differential Equations,* Marcel Dekker Inc., New York,1994.

[5] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities,* Academic Press, New York,1969.

[6] M. Eshaghi Gordji, H.Khodaei and R. Khodabakshsh, *General quartic-cube-quadratic functional equation in non-Archimedean normed spaces,* UPB Scientific Bulletin , Series A: Appl. Math. Phys.72 (2010) 69-84.

[7] M.E. Gordji, *Nearly ring homomorphism's and nearly ring derivations on non-Archimedean Banach algebras,* Abst. Appl.Anal.2010, 93247.

[8] M. Eshaghi Gordji, M.B.Ghaemi and H. Majani , *Generalised Hyers-Ulam-Rassias theorem in Menger Probabilistic normed spaces,* Discrete Dyn. Nature Society, 2010,162371.

[9] M. E.Gordji and H.Khodaei , *The fixed point method for fuzzy approximation of a Functional equation associated with inner*

product spaces, Discrete Dyn. Nature Society, 2010,140767.

[10] M. E.Gordji, *Stability of a functional equation deriving from quartic and additive functions,* Bull. Korean Math. Soci.47 (2010) 491-502.

[11] M. E.Gordji and M.S.Ghobadipour, *Stability of* (α, β, γ) *-derivations on lie C*- algebras,* Int.J.Geom.Method. Modern Phys.7(2010)1093-1102.

[12] M.E.Gordji and M.S.Moslehian , *A trick for investigation of appropriate derivations,* Mathematical Communications, 15(2010) 99-105.

[13] M.A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations,* Translated from the Russian, Macmillan, New York,1964.

[14] G.S.Ladde and V.Lakshmikantham, *A general theorem on selectors,* Bull. Acad.Polons, Sci. Ser. Math. Sci. Astr. Phys. 13 (1965) 397-403.

[15] S.Shagholi, M. Bavand Savadkouhi and M.E.Gordji, *Nearly ternary cubic homomorphism in ternary Frchet algebra,* J.Comput. Anal. Appl. 13 (2011) 1106- 1114.

[16] S.Shagholi, M.E.Gordji and M. Bavand Savadkouhi, *stability of ternary derivation on ternary Banach algebras,* J.Comput. Anal. Appl. 13 (2011) 1097-1105.